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TORSION-FREE COVERS AND PURE-INJECTIVE ENVELOPES OVER VALUATION DOMAINS

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ABSTRACT

We describe the torsion-free covers of cyclic modules, the pure-injective envelopes of ideals, the maximal immediate extensions of localizations and the injective envelopes of cyclics over valuation domains. We study the relations among these modules. This paper generalizes some results of Banaschewski, Cheatham, Enochs and Nishi.

The purpose of this paper is to describe some relations among the torsion-free covers of cyclic modules, the pure-injective envelopes of ideals, the maximal immediate extensions of localizations and the injective envelopes of cyclics when the ring R under consideration is a valuation domain. The pure-injective envelope of an ideal I of R is described as the ideal generated by I in a maximal immediate extension of the localization of R at a suitable prime ideal determined by I. The pure-injective envelope of I and the injective envelope $E(R/I)$ of the cyclic module R/I are found to be correspondent in Matlis' equivalence between torsion-free cotorsion modules and torsion h-divisible modules. The endomorphism ring of the pure-injective envelope of I is found to be exactly the endomorphism ring of $E(R/I)$, or, alternatively, the R-endomorphism ring of any maximal immediate extension of the localization of R at the prime ideal determined by I. We are able to derive a description of the torsion-free covers of the uniserial modules that are homomorphic images of R-submodules of the field of fractions of R. Finally we give various characterizations of valuation domains one of whose localization is maximal or almost maximal making use of the pure-injectivity of ideals, the torsion-free covers and the vanishing of homological functors.

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The content of this paper also improves some results of [1], [2], [3] and [11].

1. Preliminaries

A *valuation domain* R is a commutative integral domain such that for any two elements r and s of R, either r divides s or s divides r. Equivalently, for any two ideals I and J of R, either $I \leqq J$ or $J \leqq I$. We will assume that the valuation domain R is not a field and denote the field of fractions of R by Q. An ideal I of R is *proper* if $I \neq R$.

Let R and S be valuation domains and S an extension of R and let the maximal ideal of R be M. Then S is an *immediate extension* if the natural homomorphism $R/M \rightarrow S/MS$ is an isomorphism and if there is a one-to-one correspondence between the ideals of R and S given by $I \rightarrow IS$ and $J \rightarrow J \cap R$ for every ideal I of R and J of S. R is a *maximal valuation domain* if it has no proper immediate extension. Any valuation domain R has an immediate extension $\tilde{R}$ which is maximal. The ring $\tilde{R}$ is not in general unique as a ring [6], but its R-module structure is unique [12, Cor. 9]. The R-algebra $\tilde{R}$ is flat.

An *almost maximal valuation domain* R is a valuation domain whose completion $\hat{R}$ is a maximal valuation domain. The valuation domain R is almost maximal if and only if Q/R is an injective R-module [7, Th. 4].

If R is a valuation domain, a submodule A of an R-module B is *pure* in B if $rA = A \cap rB$ for any $r \in R$. If B is torsion-free, A is pure in B if and only if B/A is torsion-free. A module C is *pure-injective* if any homomorphism $A \rightarrow C$ can be extended to a homomorphism $B \rightarrow C$ whenever A is a pure submodule of B. If A is a pure submodule of B, then B is a *pure-essential extension* of A if there are no nonzero submodules $S \leqq B$ with $S \cap A = 0$ and $A + S/S$ pure in B/S. A pure extension B of A is a *pure-injective envelope* if B is pure-injective and the extension is pure-essential. Pure-injective envelopes exist and are unique up to isomorphism [12, Prop. 6]. The symbol $\mathrm{PE}_R(A)$ will denote a pure-injective envelope of the R-module A. If $\tilde{R}$ is a maximal immediate extension of R, then $\tilde{R} \cong \mathrm{PE}_R(R)$ [12, Prop. 13]. In particular the valuation domain R is maximal if and only if R is pure-injective as an R-module.

A module A is *h-reduced* if $\mathrm{Hom}_R(Q, A) = 0$ [8]. Warfield gave a complete set of invariants for the torsion-free, h-reduced, pure-injective modules over a valuation domain. Because of the fundamental importance of Warfield's result to this paper, we state his theorem.

THEOREM A (Warfield [12, Th. 5]). *A torsion-free, h-reduced, pure-injective module over a valuation domain R is the pure-injective envelope of a direct sum of* *ideals, and the number of ideals of each isomorphism type form a complete set of invariants.*

(If C is the module, the direct sum of ideals is any submodule B of C which is a direct sum of ideals, which is pure in C and which is maximal with respect to these two properties.)

The injective envelope of an R-module A is denoted $E_R(A)$. The *R-topology* on A is defined by letting the submodules of A of the form rA, where $r \in R$ and $r \neq 0$, be a base for the open neighborhoods of 0 in A [8]. All complete modules are assumed to be Hausdorff. A module C is a *cotorsion module* if $\mathrm{Hom}_R(Q, C) = 0$ and $\mathrm{Ext}^1_R(Q, C) = 0$. An h-reduced, torsion-free R-module is cotorsion if and only if it is complete in the R-topology [8, Th. 9]. If C is a cotorsion module, then $C \cong \mathrm{Ext}^1_R(Q/R, C)$.

Let A be an R-module, D a torsion-free R-module, and $\theta : D \to A$ an R-homomorphism. The pair (D, θ) is a *torsion-free lifting* of A if given any torsion-free R-module X and R-homomorphism $f : X \to A$, then there exists an R-homomorphism $\lambda : X \to D$ such that $\theta\lambda = f$. A torsion-free lifting (D, θ) of A is called a *torsion-free cover* of A if $\ker\theta$ contains no nonzero pure R-submodule of D. Sometimes we shall also call torsion-free cover of A the module D or the mapping $\theta : D \to A$. We state Banaschewski's construction of torsion-free covers without proof.

THEOREM B (Banaschewski [1]). *Let A be an R-module and $E(A)$ its injective envelope. Let $T = \{f \in \mathrm{Hom}_R(Q, E(A)) \mid f(1) \in A\}$ and define $\phi : T \to A$ by $\phi(f) = f(1)$ for all $f \in T$. Then (T, ϕ) is a torsion-free cover of A.*

If (D, θ) is a torsion-free lifting of A, then there exists a splitting monomorphism $\lambda : T \to D$ such that $\phi = \theta\lambda$ and $\ker\theta$ contains a complement of $\lambda(T)$ in D. Torsion-free covers are unique up to isomorphism.

2. Pure-injective envelopes of ideals

Recall that a family $\{M_\lambda \mid \lambda \in \Lambda\}$ of submodules of a module M is *independent* if its sum is direct, $\Sigma_\lambda M_\lambda = \bigoplus_\lambda M_\lambda$. A *maximal independent family* $\{M_\lambda \mid \lambda \in \Lambda\}$ of submodules of M is an independent family that is not properly contained in any other independent family; equivalently, the direct sum $\bigoplus_\lambda M_\lambda$ is an essential submodule of M.

PROPOSITION 1. *Let R be a valuation domain, $\tilde{R}$ a maximal immediate extension of R, I a nonzero proper ideal of R, $E = \{\varepsilon_\lambda \mid \lambda \in \Lambda\}$ a set of units of $\tilde{R}$,* *and let $\bar{\varepsilon}_\lambda$ denote the image of ε_λ in the canonical homomorphism $\tilde{R} \to \tilde{R}/I\tilde{R}$. The following are equivalent:*

(1) *The family $\{R\bar{\varepsilon}_\lambda \mid \lambda \in \Lambda\}$ is an independent family of submodules of $\tilde{R}/I\tilde{R}$.*

(2) *The family $\{I\varepsilon_\lambda \mid \lambda \in \Lambda\}$ is an independent family of submodules of $I\tilde{R}$ and its sum $\bigoplus_\lambda I\varepsilon_\lambda$ is pure in $I\tilde{R}$.*

PROOF. (1)$\Rightarrow$(2). The family $\{I\varepsilon_\lambda \mid \lambda \in \Lambda\}$ is independent, because if there is a linear combination $i_1\varepsilon_{\lambda_1} + \cdots + i_n\varepsilon_{\lambda_n} = 0$ where the i_j's are nonzero elements of I, dividing by a suitable i_j we obtain a linear combination $r_1\varepsilon_{\lambda_1} + \cdots + r_n\varepsilon_{\lambda_n} = 0$ where the r_j's are nonzero elements of R and one of the r_j's is 1. The reduction modulo $I\tilde{R}$ gives that the family $\{R\bar{\varepsilon}_\lambda \mid \lambda \in \Lambda\}$ is not independent.

In order to show that the sum $\bigoplus_\lambda I\varepsilon_\lambda$ is pure in $I\tilde{R}$, we must show that if $r \in R$, $r \neq 0$, $i, i_1, \ldots, i_n \in I$, $s \in \tilde{R}$, $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $ris = i_1\varepsilon_{\lambda_1} + \cdots + i_n\varepsilon_{\lambda_n} \in rI\tilde{R} \cap (\bigoplus_\lambda I\varepsilon_\lambda)$, then $ris \in r(\bigoplus_\lambda I\varepsilon_\lambda)$. Suppose r does not divide one of $i_1, \ldots, i_n$; then one of the elements $i_1/r, \ldots, i_n/r$ of the field of fractions Q is not in R. Therefore for a suitable $t \in R$, all the elements $ti_1/r, \ldots, ti_n/r$ are in R and one of them is 1. Hence in the field of fractions $Q(\tilde{R})$ of $\tilde{R}$ the equality $tis = (ti_1/r)\varepsilon_{\lambda_1} + \cdots + (ti_n/r)\varepsilon_{\lambda_n}$ holds. All the summands in this equality are in $\tilde{R}$ and the reduction modulo $I\tilde{R}$ gives the equality $(ti_1/r)\bar{\varepsilon}_{\lambda_1} + \cdots + (ti_n/r)\bar{\varepsilon}_{\lambda_n} = 0$. This contradicts the independence of the family $\{R\bar{\varepsilon}_\lambda \mid \lambda \in \Lambda\}$, because one of the coefficients is 1. Therefore r divides $i_1, \ldots, i_n$ in R and $is = (i_1/r)\varepsilon_{\lambda_1} + \cdots + (i_n/r)\varepsilon_{\lambda_n}$ in $\tilde{R}$. By reducing modulo $I\tilde{R}$, we obtain $(i_1/r)\bar{\varepsilon}_{\lambda_1} + \cdots + (i_n/r)\bar{\varepsilon}_{\lambda_n} = 0$. The independence of the family $\{R\bar{\varepsilon}_\lambda \mid \lambda \in \Lambda\}$ gives $(i_1/r)\bar{\varepsilon}_{\lambda_1} = \cdots = (i_n/r)\bar{\varepsilon}_{\lambda_n} = 0$, i.e., $i_1/r, \ldots, i_n/r \in I$. Thus $i_1\varepsilon_{\lambda_1} + \cdots + i_n\varepsilon_{\lambda_n} \in r(\bigoplus_\lambda I\varepsilon_\lambda)$, and (2) is proved.

(2)$\Rightarrow$(1). Suppose (2) holds and fix a linear combination $r_1\bar{\varepsilon}_{\lambda_1} + \cdots + r_n\bar{\varepsilon}_{\lambda_n} = 0$, $r_i \in R$. Let t be a nonzero element of I. Then $r_1\varepsilon_{\lambda_1} + \cdots + r_n\varepsilon_{\lambda_n} \in I\tilde{R}$ implies $tr_1\varepsilon_{\lambda_1} + \cdots + tr_n\varepsilon_{\lambda_n} \in tI\tilde{R} \cap (\bigoplus_\lambda I\varepsilon_\lambda) = t(\bigoplus_\lambda I\varepsilon_\lambda)$ because $\bigoplus_\lambda I\varepsilon_\lambda$ is pure in $I\tilde{R}$. Therefore all the coefficients $tr_1, \ldots, tr_n$ are in tI, so that $r_1, \ldots, r_n \in I$, and $r_1\bar{\varepsilon}_{\lambda_1} = \cdots = r_n\bar{\varepsilon}_{\lambda_n} = 0$. This proves (1).

COROLLARY 2. *Let R be a valuation domain, $\tilde{R}$ a maximal immediate extension of R, I a nonzero proper ideal of R, $E = \{\varepsilon_\lambda \mid \lambda \in \Lambda\}$ a set of units of $\tilde{R}$, and let $\bar{\varepsilon}_\lambda$ denote the image of ε_λ in the canonical homomorphism $\tilde{R} \to \tilde{R}/I\tilde{R}$. The following are equivalent:*

(1) *The family $\{R\bar{\varepsilon}_\lambda \mid \lambda \in \Lambda\}$ is a maximal independent family of submodules of $\tilde{R}/I\tilde{R}$.*

(2) *The sum $\sum_\lambda I\varepsilon_\lambda \leqq I\tilde{R}$ is a direct sum and its pure-injective envelope is $I\tilde{R}$.*

PROOF. The Corollary follows immediately from Proposition 1, Warfield's

Theorem A and the following remark: any pure submodule of $I\tilde{R}$ of rank one is equal to $I\varepsilon$ for a suitable unit $\varepsilon \in \tilde{R}$.

In Fuchs' and Salce's terminology [4], Property (2) could be stated by saying that $\Sigma_\lambda I\varepsilon_\lambda$ is a *basic submodule* of $I\tilde{R}$.

We are now ready for our first theorem. Recall that if I is a nonzero proper ideal of a valuation domain R, the *prime ideal determined by* I is $I^{\#} = \{r \in R \mid rI \neq I\}$ (see [11] or [5, §2]). If $I = 0$ set $I^{\#} = 0$. Since the elements of R not in $I^{\#}$ act as automorphisms on I, the ideal I of R is actually an ideal of $R_{I^{\#}}$, the localization of R at $I^{\#}$.

THEOREM 3. *Let I be a proper ideal of a valuation domain R, $I^{\#}$ the prime ideal determined by I, $R_{I^{\#}}$ the localization of R at $I^{\#}$, $\tilde{R}_{I^{\#}}$ a maximal immediate extension of $R_{I^{\#}}$. Then $I\tilde{R}_{I^{\#}}$ is a pure-injective envelope of the R-module I.*

REMARK. The ring $\tilde{R}_{I^{\#}}$ must not be confused with the localization of the maximal immediate extension $\tilde{R}$ of R at the prime ideal $I^{\#}$, which is a (proper) overring of $\tilde{R}_{I^{\#}}$.

PROOF. If $I = 0$ the statement is trivial. Suppose $I \neq 0$. The R-submodule $I = IR_{I^{\#}}$ of $I\tilde{R}_{I^{\#}}$ is pure, because if $r \in R$, $rI\tilde{R}_{I^{\#}} \cap I \leqq rI\tilde{R}_{I^{\#}} \cap R_{I^{\#}} = rI$. In order to prove that $I\tilde{R}_{I^{\#}}$ is a pure-injective R-module it is sufficient to prove that for any family x_λ $(\lambda \in \Lambda)$ of elements of $I\tilde{R}_{I^{\#}}$ and any family r_λ $(\lambda \in \Lambda)$ of elements of R, if the family $\{x_\lambda + r_\lambda I\tilde{R}_{I^{\#}} \mid \lambda \in \Lambda\}$ has the finite intersection property, then it has nonempty intersection [12, Th. 4]. But $x_\lambda + r_\lambda I\tilde{R}_{I^{\#}}$ is a coset of the $\tilde{R}_{I^{\#}}$-module $\tilde{R}_{I^{\#}}$, which is linearly compact in the discrete topology because $\tilde{R}_{I^{\#}}$ is a maximal valuation domain. Therefore $I\tilde{R}_{I^{\#}}$ is pure-injective.

In order to show that $I\tilde{R}_{I^{\#}}$ is the pure-injective envelope of I, by Theorem A it is now sufficient to prove that $I\tilde{R}_{I^{\#}}$ does not contain R-submodules J of rank 1 such that $I \oplus J$ is pure in $I\tilde{R}_{I^{\#}}$. Assume that it contains such a submodule J. If $t \in R$ and $t \notin I^{\#}$, then $tJ = J \cap t(I\tilde{R}_{I^{\#}}) = J \cap I\tilde{R}_{I^{\#}} = J$, i.e., J is an $R_{I^{\#}}$-submodule of $I\tilde{R}_{I^{\#}}$, and it is $R_{I^{\#}}$-pure of rank 1. As we have already remarked in the proof of Corollary 2, $J = Iu$ for some unit $u \in \tilde{R}_{I^{\#}}$. Now apply Proposition 1 to the units 1 and u of $\tilde{R}_{I^{\#}}$. Since $I \oplus Iu$ is pure in $I\tilde{R}_{I^{\#}}$, we have that $R_{I^{\#}}\bar{1} \cap R_{I^{\#}}\bar{u} = 0$, where $\bar{1}, \bar{u} \in \tilde{R}_{I^{\#}}/I\tilde{R}_{I^{\#}}$. Since $\tilde{R}_{I^{\#}}$ is an immediate extension of $R_{I^{\#}}$, u can be written as $u = v + rz$ with v a unit in $R_{I^{\#}}$, $r \in I^{\#}$ and $z \in \tilde{R}_{I^{\#}}$. Since $R_{I^{\#}}\bar{1} \cap R_{I^{\#}}\bar{u} = 0$, r must be nonzero; moreover $r \in I^{\#}$ implies that the ideal rI is properly contained in I, so that I is properly contained in $R \cap r^{-1}I$. Let $a \in R$, $a \in r^{-1}I$, $a \notin I$. Then $au = av + arz \in av + I\tilde{R}_{I^{\#}}$, that is, $a\bar{u} = av\bar{1}$. Since

$R_{I^\#}\bar{1} \cap R_{I^\#}\bar{u} = 0$, $a\bar{u} = av\bar{1} = 0$, contradiction because $a \notin I$. This proves that $I\tilde{R}_{I^\#}$ is the pure-injective envelope of I.

Our second theorem relates the structures of the pure-injective envelope of the ideal I and the injective envelope of the cyclic module R/I. Injective modules over valuation domains have been described by Matlis [7].

THEOREM 4. *Let I denote a nonzero proper ideal of the valuation domain R, $I^\#$ the prime ideal determined by I, $\tilde{R}_{I^\#}$ a maximal immediate extension of the localization $R_{I^\#}$ of R at $I^\#$, Q and $Q(\tilde{R}_{I^\#}) \cong Q \otimes_R \tilde{R}_{I^\#}$ the fields of fractions of R and $\tilde{R}_{I^\#}$ respectively, $\mathrm{PE}_R(I)$ a pure-injective envelope of I and $E_R(R/I)$ an injective envelope of R/I. Then the three modules $E_R(R/I)$, $(Q/R) \otimes_R \mathrm{PE}_R(I)$ and $Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#}$ are isomorphic.*

PROOF. If we apply the functor $-\otimes_R \mathrm{PE}_R(I) \cong -\otimes_R I\tilde{R}_{I^\#}$ to the exact sequence $0 \to R \to Q \to Q/R \to 0$ we obtain an exact sequence $0 \to I\tilde{R}_{I^\#} \to Q \otimes_R I\tilde{R}_{I^\#} \to (Q/R) \otimes_R \mathrm{PE}_R(I) \to 0$ because $I\tilde{R}_{I^\#}$ is torsion-free so that $\mathrm{Tor}_1^R(Q/R, I\tilde{R}_{I^\#}) = 0$. But $Q \otimes_R I\tilde{R}_{I^\#} \cong Q(\tilde{R}_{I^\#})$. Hence $(Q/R) \otimes_R \mathrm{PE}_R(I) \cong Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#}$.

Now $\mathrm{PE}_R(I) \cong I\tilde{R}_{I^\#}$ is a torsion-free, reduced, pure-injective R-module. In particular it is complete in the R-topology by [12, Th. 4], whence it is a cotorsion module [8, Th. 9]. Moreover $\mathrm{PE}_R(I)$ is an indecomposable module by Theorem A. Hence $(Q/R) \otimes_R \mathrm{PE}_R(I)$ is indecomposable [8, Th. 7]. Therefore $Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#}$ is an indecomposable R-module.

Since $\tilde{R}_{I^\#}$ is a maximal valuation domain, $Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#}$ is the injective envelope of the $\tilde{R}_{I^\#}$-module $\tilde{R}_{I^\#}/I\tilde{R}_{I^\#}$ [7, Th. 4]. But $\tilde{R}_{I^\#}$ is a flat $R_{I^\#}$-algebra and $R_{I^\#}$ is a flat R-algebra, so that $\tilde{R}_{I^\#}$ is a flat R-algebra and every $\tilde{R}_{I^\#}$-injective module is R-injective. Thus $Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#}$ is an injective R-module. Since it is indecomposable and $R/I \leqq \tilde{R}_{I^\#}/I\tilde{R}_{I^\#}$, it follows that $Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#} \cong E_R(R/I)$.

Theorem 4 has an interesting consequence. Recall Matlis' equivalence between the category of torsion h-divisible modules and the category of torsion-free cotorsion modules given by the two functors $\mathrm{Hom}_R(Q/R, -)$ and $Q/R \otimes_R -$ [8, Th. 6]. In the proof of Theorem 4 we already remarked that $\mathrm{PE}_R(I)$ is a torsion-free cotorsion module. Since $E(R/I) \cong (Q/R) \otimes_R \mathrm{PE}_R(I)$ by Theorem 4, the torsion h-divisible module $E(R/I)$ corresponds to the torsion-free cotorsion module $\mathrm{PE}_R(I)$.[†] In particular $\mathrm{PE}_R(I) \cong \mathrm{Hom}_R(Q/R, E(R/I))$. This fact and Theorem 3 give the next Corollary.

[†] *Note added in proof.* This fact was first observed by R. B. Warfield in an unpublished paper [*Relatively injective modules*, 1969, Corollary 2]. Also see R. B. Warfield, *Decompositions of injective modules*, Pacific J. Math. **31** (1969). 263–276, Proof of Corollary 4.4.

COROLLARY 5. *In the notation of Theorem* 4, *the three modules* $\mathrm{PE}_R(I)$, $\mathrm{Hom}_R(Q/R, E(R/I))$ *and* $I\tilde{R}_{I^\#}$ *are isomorphic.*

We remark in passing that $\mathrm{PE}_R(I) \cong \mathrm{Ext}^1_R(Q/R, \mathrm{PE}_R(I))$, because $\mathrm{PE}_R(I)$ is a cotorsion R-module. Moreover Matlis' equivalence gives natural R-module isomorphisms

$$\mathrm{Hom}_R(\mathrm{PE}(I), A) \cong \mathrm{Hom}_R(E(R/I), Q/R \otimes_R A),$$

$$\mathrm{Hom}_R(A, \mathrm{PE}(I)) \cong \mathrm{Hom}_R(Q/R \otimes_R A, E(R/I))$$

for every torsion-free R-module A complete in its R-topology, and

$$\mathrm{Hom}_R(B, E(R/I)) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(Q/R, B), \mathrm{PE}(I)),$$

$$\mathrm{Hom}_R(E(R/I), B) \cong \mathrm{Hom}_R(\mathrm{PE}(I), \mathrm{Hom}_R(Q/R, B))$$

for every torsion, h-divisible R-module B. In particular the endomorphism rings $\mathrm{End}_R(\mathrm{PE}(I))$ and $\mathrm{End}_R(E(R/I))$ are isomorphic. This is a half of the last result of this section.

PROPOSITION 6. *In the notation of Theorem* 4, *the three rings* $\mathrm{End}_R(\mathrm{PE}(I))$, $\mathrm{End}_R(E(R/I))$ *and* $\mathrm{End}_R(\tilde{R}_{I^\#})$ *are isomorphic.*

PROOF. By Theorem 3 it is sufficient to prove that $\mathrm{End}_R(I\tilde{R}_{I^\#}) \cong \mathrm{End}_R(\tilde{R}_{I^\#})$. Since every endomorphism of $I\tilde{R}_{I^\#}$ or $\tilde{R}_{I^\#}$ extends in a unique way to an endomorphism of their injective envelope $Q(\tilde{R}_{I^\#})$, we only have to show that for any $f \in \mathrm{End}_R(Q(\tilde{R}_{I^\#}))$ one has $f(\tilde{R}_{I^\#}) \leqq \tilde{R}_{I^\#}$ if and only if $f(I\tilde{R}_{I^\#}) \leqq I\tilde{R}_{I^\#}$. The "only if" part is trivial. To see the converse we fix $x \in \tilde{R}_{I^\#}$. Then

$$I\tilde{R}_{I^\#} f(x) = \tilde{R}_{I^\#} f(Ix) \leqq \tilde{R}_{I^\#} I\tilde{R}_{I^\#} = I\tilde{R}_{I^\#},$$

i.e., the multiplication by the element $f(x) \in Q(\tilde{R}_{I^\#})$ is an $\tilde{R}_{I^\#}$-endomorphism of $I\tilde{R}_{I^\#}$.

Now we claim that if y is any element of the maximal ideal $I^\# \tilde{R}_{I^\#}$ of $\tilde{R}_{I^\#}$, then $I\tilde{R}_{I^\#} y < I\tilde{R}_{I^\#}$. Since $y = ru$ with $r \in I^\#$ and u a unit of $\tilde{R}_{I^\#}$, we have $rIR_{I^\#} = rI < I = IR_{I^\#}$, so that

$$I\tilde{R}_{I^\#} y = I\tilde{R}_{I^\#} ru = rI\tilde{R}_{I^\#} = (rIR_{I^\#})\tilde{R}_{I^\#} < (IR_{I^\#})\tilde{R}_{I^\#} = I\tilde{R}_{I^\#}.$$

This proves the claim. Therefore the prime ideal determined by the ideal $I\tilde{R}_{I^\#}$ of $\tilde{R}_{I^\#}$ is $I^\# \tilde{R}_{I^\#}$. It follows from [5, Lemma 2] that the multiplication by an element $z \in Q(\tilde{R}_{I^\#})$ is an $\tilde{R}_{I^\#}$-endomorphism of $I\tilde{R}_{I^\#}$ if and only if $z \in \tilde{R}_{I^\#}$. In particular $f(x) \in \tilde{R}_{I^\#}$. Thus $f(\tilde{R}_{I^\#}) \leqq \tilde{R}_{I^\#}$.

Remark that part of Theorem 4 and Proposition 6 has already been obtained with different techniques by Nishi [11]. He proved that $E_R(R/I) \cong E_{R_{I^\#}}(R_{I^\#}/I) \cong E_{\tilde{R}_{I^\#}}(\tilde{R}_{I^\#}/I\tilde{R}_{I^\#})$ and $\mathrm{End}_R(E(R/I)) \cong \mathrm{End}_R(\tilde{R}_{I^\#})$.

3. Torsion-free covers of cyclic modules

In the second section we studied the relation between the pure-injective envelope of an ideal I of a valuation domain R and the injective indecomposable module $E(R/I)$. In this section we study the relation between the pure-injective envelope of I and the torsion-free cover of the cyclic R-module R/I.

THEOREM 7. *Let R be a valuation domain, I a nonzero proper ideal of R, $I^\#$ the prime ideal determined by I, $\tilde{R}_{I^\#}$ a maximal immediate extension of the localization $R_{I^\#}$ of R at $I^\#$, Q and $Q(\tilde{R}_{I^\#})$ the fields of fractions of R and $\tilde{R}_{I^\#}$ respectively, $\mathrm{PE}_R(I) = I\tilde{R}_{I^\#} \leqq Q(\tilde{R}_{I^\#})$ the pure-injective envelope of I, $E_R(R/I)$ the injective envelope of R/I and J an R-submodule of Q properly containing I. Then:*

(1) $J + \mathrm{PE}_R(I) = J + I\tilde{R}_{I^\#}$ *is the torsion-free cover of* J/I.

(2) $Q \otimes_R \mathrm{PE}_R(I) \cong Q(\tilde{R}_{I^\#})$ *is the torsion-free cover of* $E_R(R/I)$.

REMARK. The homomorphisms of the torsion-free covers are the obvious ones $J + I\tilde{R}_{I^\#} \to J + I\tilde{R}_{I^\#}/I\tilde{R}_{I^\#} \cong J/J \cap I\tilde{R}_{I^\#} = J/I$ and $Q(\tilde{R}_{I^\#}) \to Q(\tilde{R}_{I^\#})/I\tilde{R}_{I^\#} \cong E_R(R/I)$.

PROOF. (1). If X is any torsion-free R-module, any exact sequence $0 \to \mathrm{PE}_R(I) \to Y \to X \to 0$ is pure, hence it splits because $\mathrm{PE}_R(I)$ is pure-injective. Therefore $\mathrm{Ext}^1_R(X, \mathrm{PE}_R(I)) = 0$. If we apply the functor $\mathrm{Hom}_R(X, -)$ to the exact sequence $0 \to \mathrm{PE}_R(I) \to J + \mathrm{PE}_R(I) \to J/I \to 0$, we obtain an exact sequence $\mathrm{Hom}_R(X, J + \mathrm{PE}_R(I)) \to \mathrm{Hom}_R(X, J/I) \to 0$. This proves that $J + \mathrm{PE}_R(I)$ is a torsion-free lifting of J/I.

By Banaschewski's Theorem B, $J + \mathrm{PE}_R(I) = T(J/I) \oplus D$ where $T(J/I)$ is a torsion-free cover of J/I and $D \leqq \mathrm{PE}_R(I)$. The modular identity gives $\mathrm{PE}_R(I) = [T(J/I) \cap \mathrm{PE}_R(I)] \oplus D$. But $\mathrm{PE}_R(I)$ is indecomposable by Theorem A, and one of these direct summands must be zero. If $T(J/I) \cap \mathrm{PE}_R(I) = 0$, then

$$T(J/I) \cong T(J/I) + \mathrm{PE}_R(I)/\mathrm{PE}_R(I) = J + \mathrm{PE}_R(I)/\mathrm{PE}_R(I) \cong J/I,$$

contradiction because $T(J/I)$ is torsion-free and J/I is a nonzero torsion module. Thus $D = 0$ and $J + \mathrm{PE}_R(I) = T(J/I)$.

(2). By Theorem B it is clear that the torsion-free cover of R/I is an essential submodule of the torsion-free cover of $E(R/I)$, and the torsion-free cover of $E(R/I)$ is $\mathrm{Hom}_R(Q, E(R/I))$, i.e., a torsion-free divisible module. Therefore the torsion-free cover of $E(R/I)$ is the divisible envelope (= injective envelope) of the torsion-free cover $R + \mathrm{PE}_R(I) = R + I\tilde{R}_{I^\#}$ of R/I, i.e., it is isomorphic to $Q \otimes_R \mathrm{PE}(I) \cong Q(\tilde{R}_{I^\#})$.

Banaschewski proved Theorem 7 for the case of a Noetherian valuation domain R and $J = R$ [1, Prop. 6]. Enochs proved it for the case of an arbitrary valuation domain R but for I its maximal ideal and $J = R$ [3, Prop. 5.1]. Theorem 7 generalizes their results.

Corollary 8. *In the notations of Theorem* 7, $R + \mathrm{PE}_R(I)$ *and* $Q + \mathrm{PE}_R(I)$ *are the torsion-free covers of* R/I *and* Q/I *respectively. If* $\tilde{R}$ *is a maximal immediate extension of* R, *then* $Q + \tilde{R}$ *is the torsion-free cover of* Q/R.

Proof. The Corollary follows immediately from Theorem 7 by noting that the prime ideal determined by a principal ideal is the maximal ideal M of R and $Q + M\tilde{R} = Q + R + M\tilde{R} = Q + \tilde{R}$.

In particular $\mathrm{PE}_R(I)$ is the kernel of the torsion-free cover $R + \mathrm{PE}_R(I) \to R/I$; but then $\mathrm{PE}_R(I)$ is not only a cotorsion module, as we repeatedly remarked in Section 2, but it is even a *strongly cotorsion module*, i.e., $\mathrm{Ext}^n_R(Q, \mathrm{PE}_R(I)) = 0$ for all $n \geqq 0$. This follows from [3, §2].

Note that R/I is a torsion module of bounded order, and therefore it is a cotorsion module. Then [10, Prop. 3.2(2)] gives that there exists an exact sequence

$$0 \to \mathrm{Hom}_R(Q/R, E(R/I)) \to \mathrm{Hom}_R(Q/R, E(R/I)/(R/I)) \to R/I \to 0$$

that is a torsion-free cover of R/I. By the uniqueness stated in Theorem B we refind the isomorphism $\mathrm{PE}_R(I) \cong \mathrm{Hom}_R(Q/R, E(R/I))$ of Corollary 5.

Our next proposition generalizes Cheatham's Theorem 4 in [2] and some results of Enochs' [3, §§5 and 6]. It is closely related to results proved by Matlis [9] and [7, Th. 9].

Proposition 9. *Let* R *be a valuation domain,* I *a proper ideal of* R, $I^\#$ *the prime ideal determined by* I, $R_{I^\#}$ *the localization of* R *at* $I^\#$ *and* Q *the field of fractions of* R. *Then the following statements are equivalent*:

(1) $R_{I^\#}$ *is a maximal valuation domain.*

(2) I *is a pure-injective* R*-module.*

(3) *The canonical projection* $R \to R/I$ *is a torsion-free cover.*

(4) *The canonical projection* $Q \to Q/I$ *is a torsion-free cover.*

(5) $\mathrm{Ext}^1_R(A, I) = 0$ *whenever* A *is a torsion-free* R*-module.*

Proof. The equivalences are trivial for $I = 0$; suppose $I \neq 0$.

Statements (2), (3) and (4) are all equivalent to $\mathrm{PE}_R(I) = I$ by Theorem 7. Moreover $R_{I^\#}$ is maximal if and only if $R_{I^\#} = \tilde{R}_{I^\#}$, i.e., if and only if $I = I\tilde{R}_{I^\#}$ ("only if": $I = IR_{I^\#} = I\tilde{R}_{I^\#}$; "if": $x \in \tilde{R}_{I^\#}$ implies $tx \in I\tilde{R}_{I^\#} = I$ for some nonzero $t \in I$; therefore $x = r/t$ for some $r \in I$, so that $x \in Q \cap \tilde{R}_{I^\#} = R_{I^\#}$).

Hence $R_{I^\#}$ is maximal if and only if $I = \mathrm{PE}_R(I)$ by Theorem 3, and the first four statements are equivalent.

(1)$\Rightarrow$(5). If $R_{I^\#}$ is maximal, $\mathrm{Ext}^1_{R_{I^\#}}(B, I) = 0$ for B any torsion-free $R_{I^\#}$-module [7, Th. 9]. $\mathrm{Ext}^1_R(A, I) \cong \mathrm{Ext}^1_{R_{I^\#}}(A \otimes R_{I^\#}, I) = 0$ for A any torsion-free R-module [8, Th. A3].

(5)$\Rightarrow$(3). If $\mathrm{Ext}^1_R(A, I) = 0$ for any torsion-free module A, the morphism $R \to R/I$ induces an epimorphism $\mathrm{Hom}(A, R) \to \mathrm{Hom}(A, R/I)$, i.e., $R \to R/I$ is a torsion-free lifting. But R is indecomposable, so that $R \to R/I$ is a torsion-free cover by Theorem B.

When the equivalent conditions of Proposition 9 hold, that is, when $R \to R/I$ is a torsion-free cover of R/I. Matlis defines a prime ideal of R determined by I [9, §4]. This prime ideal is exactly our prime ideal $I^\#$.

For the sake of completeness we conclude by considering almost maximal valuation domains. Most of the equivalences in the next proposition have been proved by Matlis [10, Prop. 3.7, 3.8, 4.3].

PROPOSITION 10. *Let R be a valuation domain, I a nonzero proper ideal of R, $I^\#$ the prime ideal determined by I, $R_{I^\#}$ the localization of R at $I^\#$, $\hat{R}$, $\hat{I}$ the completions of R, I resp. in their R-topologies, $E_R(R/I)$ the injective envelope of R/I, and Q, $Q(\hat{R})$ the fields of fractions of R, $\hat{R}$ respectively. Then the following statements are equivalent*:

(1) *$R_{I^\#}$ is an almost maximal valuation domain.*

(2) *$\hat{I}$ is the pure-injective envelope of I.*

(3) *The canonical projection $\hat{R} \to R/I$ is a torsion-free cover of R/I.*

(4) *$E_R(R/I) \cong Q/I$.*

(5) *I has injective dimension 1.*

(6) *If $\phi : Q(\hat{R}) \to Q/I$ is the (well-defined) mapping that associates an element $q + I \in Q/I$ such that $\hat{q} - q \in I\hat{R}$ to any element $\hat{q} \in Q(\hat{R})$, ϕ is a torsion-free cover of Q/I.*

PROOF. (1) $\Rightarrow$ (2). If $R_{I^\#}$ is almost maximal, then $\tilde{R}_{I^\#} = \hat{R}_{I^\#}$ (the completion of $R_{I^\#}$), so that $\mathrm{PE}_R(I) = I\hat{R}_{I^\#}$ by Theorem 3. By [8, Th. 13(1)] $I\hat{R}_{I^\#}$ is the completion of I in the $R_{I^\#}$-topology, and by [8, Th. 15] the R-topology and the $R_{I^\#}$-topology on I coincide. Thus $\mathrm{PE}_R(I) = \hat{I}$.

(2) $\Rightarrow$ (3). By Theorem 7 $R + \hat{I}$ is a torsion-free cover of R/I. By [8, Th. 13(3)] $R + \hat{I} = \hat{R}$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). [10, Prop. 3.8].

(5) $\Rightarrow$ (6). By [10, Prop. 3.7] $\phi : Q(\hat{R}) \to Q/I$ is a torsion-free lifting of Q/I. By Theorem B, $Q(\hat{R}) = T(Q/I) \oplus D$ where $D \leqq \ker \phi = I\hat{R}$ and $T(Q/I)$ is a torsion-free cover of Q/I. Now $Q(\hat{R})$ is divisible, so that its summand D is divisible. But $D \leqq I\hat{R}$, which is h-reduced. Thus $D = 0$ and $Q(\hat{R})$ is a torsion-free cover of Q/I.

(6) $\Rightarrow$ (1). By [10, Prop. 3.7] (6) implies that I has injective dimension 1, so that Q/I is an injective R-module. Therefore Q/I is a linearly compact R-module in the discrete topology, in particular a linearly compact R_{I^*}-module. Therefore its homomorphic image Q/R_{I^*} is a linearly compact R_{I^*}-module. Thus R_{I^*} is almost maximal by [7].

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